Mock exam sets October 2024

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All questions are independent and count equally so make sure you try each of them. Good luck!

0. Prove that for any sets A, B and C we have:

$$((A \times B) \cap (C \setminus B)) \setminus ((B \times C) \cap A^2) \subseteq C$$

1. **Explain why the following argument is invalid:** Define a function $f : \mathbb{Q}^2 \to \mathbb{Q}$ by $f(\frac{p}{q}, \frac{r}{s}) = (p - s)(q - r)$. The function f is surjective so $0 = f(\frac{2}{1}, \frac{1}{2}) = f(\frac{4}{2}, \frac{1}{2}) = 2$ so $0 = 2$.

2. Suppose $f : X \to X$ is a function from the uncountable set X to itself such that $f \circ f \circ f = \mathrm{id}_X$. Show that f is invertible.

3. For $n > 0$ and non-empty sets $A_0, A_1, A_2, \ldots A_n$ define $X = \bigcup_{i=0}^{n-1} A_i$ and a function $f : X \to [n]$ by $f(x) = \#\{u \in [n] : x \in A_u\}$. Prove that for every $x \in X$ we have $f(x) > 0$.

4. Define a function $f : \{a, b, c, d, e\} \to [5]$ by $f(a) = f(c) = f(e) = 0$ and $f(b) = 1$, $f(d) = 2$. Write down explicitly what are the elements of the inverse image $f^{-1}(\{0, 2, 4\})$ and the image $f(\{a, a, a, b, e\})$.

5. Define a relation $\sim$ on the set $[3]^{[3]}$ by $f \sim g$ iff $f \circ g$ is injective. Is $\sim$ an equivalence relation? If yes, write down a system of representatives. If no, explain which property fails.

6. Explicitly write down an infinite set of sets $\mathcal{S}$ such that no two elements of $\mathcal{S}$ have the same cardinality.

7. Use induction to prove that for all positive integers n the set $\mathbb{N}^n$ is countably infinite.